APPENDIX F

Vectors, Matrices, and Group Theory

We collect here some of the most basic definitions and properties of real, finite-dimensional vectors and matrices. We intentionally ignore all of the subtleties regarding the precise definitions of vectors and tensors and supply only the "bare necessities". Many comments on the generalization of these ideas to complex vectors and infinite dimensional spaces are given in the text. We then briefly describe some of the rudiments of group theory.

F.1 Vectors and Matrices

We will take vectors to be ordered N-tuples of numbers, for example,

$$\mathbf{x} = (x_1, x_2, \ldots, x_N) \quad \text{and} \quad \mathbf{y} = (y_1, y_2, \ldots, y_N) \tag{F.1}$$

along with a *dot*- or *inner-product* of the form

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{N} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N \tag{F.2}$$

The *norm* (or generalized length) of the vector is taken to be

$$|x| = \sqrt{\mathbf{x} \cdot \mathbf{x}} \tag{F.3}$$

A *matrix* will be defined to be a square $N \times N$ array of the form

$$\mathsf{M} = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & \cdots & M_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} \end{pmatrix} \tag{F.4}$$

The *unit matrix* is given by

$$\mathsf{1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{F.5}$$

Multiplication of a vector by a matrix *on the left* (as with operators) gives a vector, that is,

$$\mathbf{x}' = \mathsf{M} \cdot \mathbf{x} \tag{F.6}$$

In component form one can write

$$(x')_i = (\mathsf{M} \cdot \mathbf{x})_i = \sum_{j=1}^{N} M_{ij} x_j \tag{F.7}$$

or more explicitly

$$\mathsf{M} \cdot \mathbf{x} = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & \cdots & M_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

$$= \begin{pmatrix} M_{11}x_1 + M_{12}x_2 + \cdots + M_{1N}x_N \\ M_{21}x_1 + M_{22}x_2 + \cdots + M_{2N}x_N \\ \vdots \\ M_{N1}x_1 + M_{N2}x_2 + \cdots + M_{NN}x_N \end{pmatrix} \tag{F.8}$$

The product of two matrices is again a matrix with the component definition

$$(\mathsf{M} \cdot \mathsf{N})_{ik} = \sum_{j=1}^{N} M_{ij} N_{jk} \tag{F.9}$$

or

$$\text{the } (ik)\text{-th element of } \mathsf{M} \cdot \mathsf{N}$$

$$\Updownarrow$$

$$(\text{the } i\text{th row of } \mathsf{M}) \text{ dotted into } (\text{the } k\text{th column of } \mathsf{N})$$

The *transpose* of the matrix M, labeled M^{T}, is obtained by “reflecting” all of its elements along the diagonal (the $i = j$ components staying fixed), that is,

$$(\mathsf{M}^{\mathrm{T}})_{ij} = (\mathsf{M})_{ji} = M_{ji} \tag{F.10}$$

so that

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \tag{F.11}$$

The generalization of this to complex matrices is the *adjoint* or *Hermitian conjugate* defined via

$$\mathsf{M}^{\dagger} = (\mathsf{M}^{\mathrm{T}})^{*} = (\mathsf{M}^{*})^{\mathrm{T}} \tag{F.12}$$

which "flips" the matrix elements *and* takes their complex conjugate.

The equivalent of the expectation value of an operator in a quantum state is given by

$$\langle x|M|x\rangle \;\sim\; \mathbf{x}\cdot\mathsf{M}\cdot\mathbf{x} = \sum_{j=1}^{N}\sum_{k=1}^{N} x_j M_{jk} x_k \tag{F.13}$$

A matrix transformation of the form Eqn. (F.6) generally changes the norm of the vector since

$$\begin{aligned} \mathbf{x}'\cdot\mathbf{x}' = \sum_i x'_i x'_i &= \sum_{i=1}^{N}\left(\sum_{j=1}^{N} M_{ij}x_j\right)\left(\sum_{k=1}^{N} M_{ik}x_k\right) \\ &= \sum_{j,k=1}^{N} x_j\left[\sum_i (M^T)_{jk}M_{ik}\right] x_k \\ &= \sum_{j,k=1}^{N} x_j P_{jk} x_k \\ &\neq \sum_j x_j x_j = \mathbf{x}\cdot\mathbf{x} \end{aligned} \tag{F.14}$$

unless one has

$$\sum_i (M^T)_{ji}M_{ik} = P_{jk} = \delta_{j,k} \quad \text{or} \quad \mathsf{M}^{\mathrm{T}}\cdot\mathsf{M} = \mathsf{P} = \mathsf{1} \tag{F.15}$$

Matrices satisfying Eqn. (F.15) are said to be *orthogonal.*

Finally, the *determinant* of a matrix is a number formed from the elements of the matrix via

$$\det(\mathsf{M}) = \sum_{i_1,i_2,\ldots,i_N=1}^{N} \epsilon_{(i_1,i_2,\ldots,i_N)} M_{1,i_1} M_{2,i_2}\cdots M_{N,i_N} \tag{F.16}$$

The *totally antisymmetric symbol*,[1] $\epsilon_{(i_1,i_2,\ldots,i_n)}$, is defined via

$$\epsilon_{(i_1,i_2,\ldots,i_N)} = \begin{cases} +1 & \text{if } (i_1, i_2, \ldots, i_N) \text{ is an} \\ & \quad \text{even permutation of} (1, 2, \ldots, N) \\ -1 & \text{if it is an odd permutation} \\ 0 & \text{otherwise} \end{cases} \tag{F.17}$$

It vanishes if any two of its indices are the same and is antisymmetric under the interchange of any pair of indices. Each term in the determinant then consists of a product of one element from each row, with appropriate signs. For example,

$$\det(\mathsf{A}) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{F.18}$$

and

$$\begin{aligned} \det(\mathsf{B}) = \det \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \\ = b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{32}b_{21} - b_{13}b_{31}b_{22} \\ - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33} \end{aligned} \tag{F.19}$$

One important property of determinants is that the interchange of any two rows (or columns) gives the same value, but with an additional factor of (-1); this follows from the definition in Eqn. (F.16) and the antisymmetry of the ϵ symbol.

Equations of the form

$$\mathsf{M} \cdot \mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda \tag{F.20}$$

are called *eigenvalue problems* and λ is the *eigenvalue* and $\mathbf{v}_\lambda$ the corresponding *eigenvector*. We can also write this as

$$(\mathsf{M} - \lambda \mathsf{1}) \cdot \mathbf{v} = 0 \tag{F.21}$$

or in matrix form as

$$\begin{pmatrix} M_{11} - \lambda & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} - \lambda & \cdots & M_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} - \lambda \end{pmatrix} = 0 \tag{F.22}$$

This is equivalent to a set of N linear equations in N unknowns and the condition for a solution to exist is that

$$\det(\mathsf{M} - \lambda \mathsf{1}) = 0 \tag{F.23}$$

[1] It is also called the Levi-Civita symbol.

and this condition determines the allowed eigenvalues λ. Real matrices for which

$$\mathsf{M}^{\mathrm{T}} = \mathsf{M} \tag{F.24}$$

are called *symmetric*, while complex matrices for which

$$\mathsf{M}^{\dagger} = \mathsf{M} \tag{F.25}$$

are called *Hermitian* and both have the properties:

- The eigenvalues of M are real.
- The eigenvectors of M corresponding to different eigenvalues are orthogonal.

Example F.1. Eigenvalues and eigenvectors of a simple matrix

The eigenvalues of the matrix

$$\mathsf{M} = \begin{pmatrix} 23 & -36 \\ -36 & 2 \end{pmatrix} \tag{F.26}$$

are determined by the condition

$$\det \begin{pmatrix} 23-\lambda & -36 \\ -36 & 2-\lambda \end{pmatrix} = \lambda^2 - 25\lambda - 1250 = 0 \tag{F.27}$$

which has solutions $\lambda_1 = 50$ and $\lambda_2 = -25$. The eigenvector corresponding to λ_1 can be found by insisting that

$$\begin{pmatrix} 23-50 & -36 \\ -36 & 2-50 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{F.28}$$

or

$$\mathbf{v}_1 = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix} \tag{F.29}$$

when normalized so that $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1$. One similarly finds that

$$\mathbf{v}_2 = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \tag{F.30}$$

(with $\mathbf{v}_2 \cdot \mathbf{v}_2 = 1$ by construction) and we confirm that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

Finally, some useful identities involving the scalar and cross-products of vectors are

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{C} \cdot \mathbf{A}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{F.31}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{F.32}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{F.33}$$

F.2 Group Theory

We conclude with a brief definition of a mathematical *group*. A set of elements $G = \{g_1, g_2, \ldots\}$ along with a binary operation (often called "group multiplication") denoted by $g_1 \cdot g_2$ constitutes a group if it satisfies four conditions:

1. The product of any two group elements is also a group element, that is, $g_3 = g_1 \cdot g_2$ is from G if g_1, g_2 are also; the group is closed under multiplication.
2. The group multiplication is associative, namely

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \tag{F.34}$$

3. There is a unique group element, labeled I or the identity element, which satisfies

$$I \cdot g_i = g_i \cdot I = g_i \tag{F.35}$$

   for all $g_i \in G$.
4. Every group element, g_i, has a unique inverse element, labeled g_i^{-1} which satisfies

$$g_i \cdot g_i^{-1} = g_i^{-1} \cdot g_i = I \tag{F.36}$$

The set of group elements can be finite or infinite. Groups for which the multiplication gives the same answer in either order, that is, for which $g_i \cdot g_j = g_j \cdot g_i$ for every pair of group elements is called a *commutative* or *Abelian group*.

F.3 Problems

PF.1. Show that $(\mathsf{A} \cdot \mathsf{B} \cdots \mathsf{Y} \cdot \mathsf{Z})^{\mathrm{T}} = \mathsf{Z}^{\mathrm{T}} \cdot \mathsf{Y}^{\mathrm{T}} \cdots \mathsf{B}^{\mathrm{T}} \cdot \mathsf{A}^{\mathrm{T}}$.

PF.2. Show that the cross-product of two vectors can be written in the form

$$\mathbf{A} \times \mathbf{B} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix} \tag{F.37}$$

PF.3. Find the eigenvalues and eigenvectors of the Hermitian matrix

$$\mathsf{M} = \begin{pmatrix} 4 & 3+2i \\ 3-2i & -5 \end{pmatrix} \tag{F.38}$$

and show explicitly that the two eigenvectors are orthogonal. Note that the dot-product of two *complex* vectors is defined via $\mathbf{v}_1^* \cdot \mathbf{v}_2$.