APPENDIX C

Complex Numbers and Functions

Because the formalism of quantum mechanics requires the manipulation of complex variables, we review here some of the basic definitions and formulae governing their properties. The imaginary unit, i, is defined[1] via $i = \sqrt{-1}$, and a general *complex number* is given by

$$z = a + ib \tag{C.1}$$

where a, b themselves have purely real values. The values a and b are called the *real* and *imaginary* parts of z, respectively; these are often written in the form

$$a = Re(z) \quad \text{and} \quad b = Im(z) \tag{C.2}$$

Complex numbers obey standard algebraic relations. For example, if $z_{1,2} = a_{1,2} + ib_{1,2}$, we have addition and subtraction given by

$$z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2) \tag{C.3}$$

while multiplication is carried out by using the distributive law, giving

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \tag{C.4}$$

More complicated functions can often be evaluated using series expansions. For example, for θ real we can write

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right) \\ e^{i\theta} &= \cos(\theta) + i\sin(\theta) \end{aligned} \tag{C.5}$$

using the series expansions in Appendix D.2.

[1] Engineers often use the notation $j = \sqrt{-1}$.

The *complex conjugate* of a complex number is defined via

$$z^* \equiv a - ib \tag{C.6}$$

that is, by letting $i \to -i$. A useful relation is

$$|z|^2 \equiv zz^* = (a+ib)(a-ib) = a^2 + b^2 \tag{C.7}$$

which defines the *modulus*, $|z|$, of a complex number via

$$|z| = \sqrt{a^2 + b^2} \tag{C.8}$$

This quantity is the analog of the absolute value of a real number. We can make use of the identity (C.5) to also write

$$a + ib = z \equiv |z|e^{i\theta} = |z|\cos(\theta) + i|z|\sin(\theta) \tag{C.9}$$

where θ is called the *phase* or *argument* of the complex number z; it is given by

$$\tan(\theta) = \frac{b}{a} = \frac{Im(z)}{Re(z)} \tag{C.10}$$

This form for complex numbers is useful as it shows that

$$|z_1 z_2| = |z_1||z_2| \tag{C.11}$$

We obviously have $z^* = |z|e^{-i\theta}$, so complex conjugation "flips the phase" of z, but keeps its modulus fixed. A complex number with $|z| = 1$, that is, of the form $z = e^{i\theta}$, is often said to be "just a phase". A general complex number can be represented as a point (or vector) in the complex plane, as shown in Fig. C.1, and addition and subtraction can be given a vector interpretation.

Some useful formulae (for θ real) are

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{C.12}$$

which are easily derived by combining Eqn. (C.5) and its complex conjugate. One also has

$$\cos(i\theta) = \cosh(\theta) \quad \text{and} \quad \sin(i\theta) = i\sinh(\theta) \tag{C.13}$$

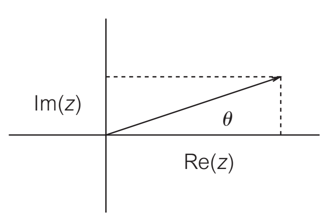

Figure C.1. Representation of a complex number, z, as a vector in the complex plane.

Other familiar trig identities are easily proved using complex notation. For example, one has

$$\begin{aligned}
2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\
&= 2\left(\frac{e^{i(\alpha+\beta)/2} - e^{-i(\alpha+\beta)/2}}{2i}\right)\left(\frac{e^{i(\alpha-\beta)/2} + e^{i(\alpha-\beta)/2}}{2}\right) \\
&= \sin(\alpha) + \sin(\beta)
\end{aligned} \tag{C.14}$$

One also has

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{C.15}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{C.16}$$

and the related special cases for double-angle formulae,

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{C.17}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{C.18}$$

or

$$\sin^2(\alpha) = \frac{1}{2}[1 - \cos(2\alpha)] \tag{C.19}$$

$$\cos^2(\alpha) = \frac{1}{2}[1 + \cos(2\alpha)] \tag{C.20}$$

C.1 Problems

PC.1. Calculate the result of dividing two complex numbers; specifically, if

$$Re(w) + iIm(w) = w = \frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \tag{C.21}$$

find explicit expressions for $\mathrm{Re}(w), \mathrm{Im}(w)$.

PC.2. Find a general expression for the modulus of $|w|$ if

$$w = \exp\left(\frac{1}{a + ib}\right) \tag{C.22}$$

where a, b are real; find a numerical value if $a = 2$ and $b = 1$.

PC.3. Verify the identity

$$|e^{i\alpha} + e^{i\beta}| = \left|2\cos\left(\frac{\alpha+\beta}{2}\right)\right| \tag{C.23}$$

if α, β are real. Derive a similar identity for $|e^{i\alpha} - e^{i\beta}|$.

PC.4. If $z = 2 - 3i$, find $\sqrt{z}$.

PC.5. Show that the following identities hold for any positive integer n,

$$e^{-\pi i n^2/2} = \left\{ \begin{matrix} +1 & \text{for } n \text{ even} \\ -i & \text{for } n \text{ odd} \end{matrix} \right\} = \left(\frac{1-i}{2}\right) + \left(\frac{1+i}{2}\right)(-1)^n$$
$$= \frac{1}{\sqrt{2}}\left[e^{-i\pi/4} + e^{+i\pi/4}(-1)^n\right] \qquad \text{(C.24)}$$